Phasors and complex numbers

Phasor $V_M \angle \phi$ is a combination of two parameters: the amplitude $V_M$ (the length of the arrow) and the phase angle $\phi$.

This pair of two parameters fully describes the AC waveform of a given angular frequency $\omega$.

A pair of two parameters: the radius $R$ and the angle $\phi$ fully describes the position of the point (red dot) on the $x-y$ plane.

The position of the point can also be described by a pair of two numbers: $x_0$ and $y_0$.

The pair $(R,j)$ or $(x_0, y_0)$ is called a complex number.
Complex numbers

Complex numbers provide a convenient technique to describe the phasors related to AC voltages and currents.

\[ N = x + jy, \quad x = Re(N); \ y = Im(N); \]

Complex number is the sum of two parts: Real part, \( Re(N) \), and “Imaginary” part, \( Im(N) \) multiplied by a so-called imaginary unit \( j \) (sometimes also denoted as \( i \)).

Imaginary unit is a special number such that \( j^2 = -1 \);

Introduction of the imaginary unit is needed to provide simple and convenient rules to operate the pairs of two number (complex numbers) as a single number.

The real number, the square of which is equal to (-1) does not exist. Therefore, \( j \) is not a real number.
Complex numbers

There are two commonly accepted notations for the $\sqrt{-1}$, $j$ or $i$

Some examples

1. \(1 + 4i\) is a complex number with a real part = 1, imaginary part = 4
2. \(2 - j2\) is a complex number with a real part = 2, imaginary part = -2.
3. \(-4i\) is a complex number with a real part = 0, imaginary part = -4
4. \(5\) is a complex number with a real part = 5, imaginary part = 0
Complex numbers

There are two equivalent ways to define the complex number: by Real and Imaginary parts (x and y coordinates) or by the radius and the angle of the arrow pointing to the number. This radius is also called a “modulus”, or an “absolute value” of the complex number.

\[ |R| = \sqrt{y_1^2 + x_1^2} \]
Complex numbers

The angle that the vector makes with the axis $x$ is called an argument or phase angle.

Complex number $N$ represented by the position of the dot on x-y plane:

$$\phi = \arctan \left( \frac{y_1}{x_1} \right) = \arctan \left( \frac{\text{Im}(R)}{\text{Re}(R)} \right)$$

$$x_1 = |R| \cos (\phi); \quad y_1 = |R| \sin (\phi)$$
Complex numbers

To summarize, any complex number can be described in two ways:

\[ N = x_1 + jy_1 \quad (Algebraic, \ or \ Cartesian \ form) \]

\[ N = |R| \angle \phi \quad (Phasor, \ or \ Polar \ form) \]
The algebra of Complex numbers:

all the algebraic operations, addition, subtraction, multiplication and division follow the algebra rules for real numbers; whenever there is a term $i^2$ or $(j^2)$ replace it with (-1)

$z_1 = a + bi$ and $z_2 = c + di$;

addition:
$z_1 + z_2 = a + bi + c + di = (a + c) + (b + d)i$

subtraction:
$z_1 - z_2 = (a - c) + (b - d)i$
The algebra of Complex numbers:

all the algebraic operations, addition, subtraction, multiplication and division follow the algebra rules for real numbers; whenever there is a term $i^2$ or $(j^2)$ replace it with $(-1)$

\[ z_1 = a + bi \quad \text{and} \quad z_2 = c + di; \]

**multiplication:**

\[ z_1z_2 = (a + bi)(c + di) = ac + adi + bci + bdi^2; \]
\[ i^2 = -1, \text{ therefore:} \]

\[ z_1z_2 = (a + bi)(c + di) = ac + adi + bci - bd \]

\[ z_1z_2 = ac - bd + (ad + bc)i \]
The algebra of Complex numbers:

all the algebraic operations, addition, subtraction, multiplication and division follow the algebra rules for real numbers; whenever there is a term $i^2$ or $(j^2)$ replace it with (-1)

$$z_1 = a + bi \quad \text{and} \quad z_2 = c + di;$$

**division**

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$$

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$
The algebra of Complex numbers: Polar → Cartesian form conversion

$N = x_1 + jy_1$ (*Algebraic, or Cartesian form*)

$|N| \angle \phi$ (*Phasor, or Polar form*)

$\text{Re}(N) = x = |N| \cdot \cos(\phi)$;

$\text{Im}(N) = y = |N| \cdot \sin(\phi)$;
The algebra of Complex numbers: Cartesian -> Polar form conversion

\[ N = x_1 + jy_1 \text{ (Algebraic, or Cartesian form)} \]

\[ |N| \quad \phi \quad \text{(Phasor, or Polar form)} \]

\[ |N| = \sqrt{x^2 + y^2} = \sqrt{\text{Re}(N)^2 + \text{Im}(N)^2} \]

\[ \tan(\phi) = \frac{y}{x} = \frac{\text{Im}(N)}{\text{Re}(N)}, \quad \text{or:} \]

\[ \phi = \arctan \left( \frac{y}{x} \right) = \arctan \left[ \frac{\text{Im}(N)}{\text{Re}(N)} \right] \]